Conformal Parameterization using Discrete Calabi Flow

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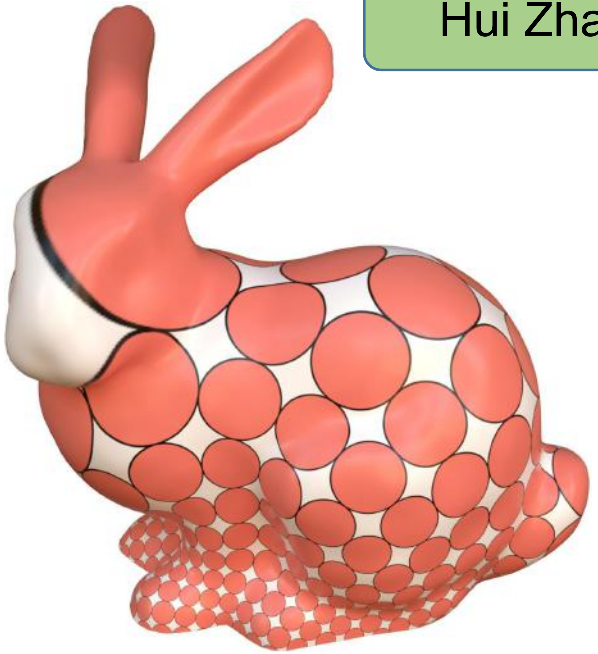
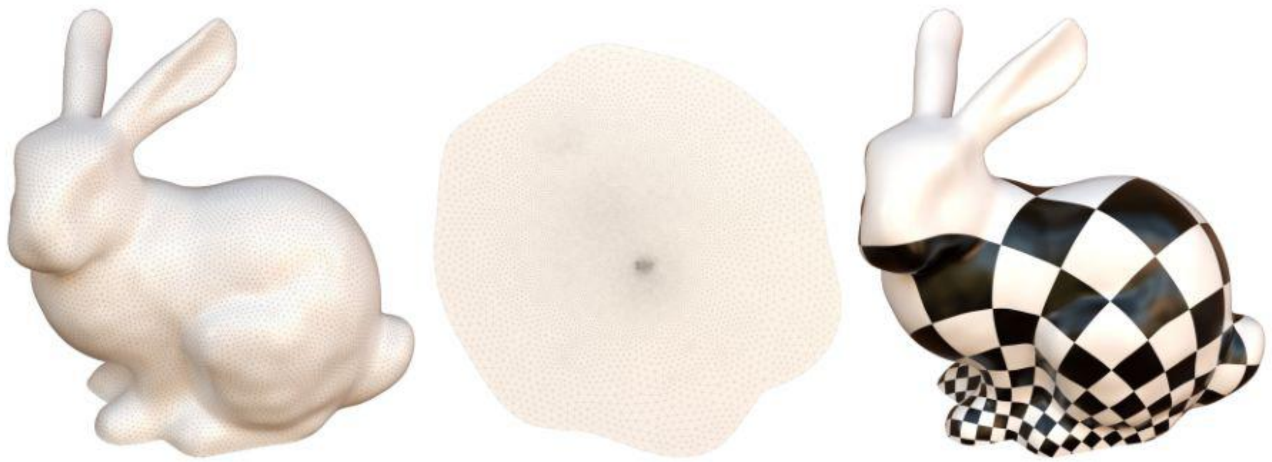


Bunny (a) and its parameterization by Calabi flow (b), and texturings.



(a) (b) (c)

(d) (e) (f)

Overview

1. The discrete Calabi flow and dual-laplacian operator
2. A different conformal parameterization algorithm
3. The similar results with Ricci flow and CETM

M. Jin, J. Kim, and X. Gu. Discrete surface ricci flow: Theory and applications. In *IMA Conference on the Mathematics of Surfaces*, 2007.

B. Springborn, P. Schröder, and U. Pinkall. Conformal equivalence of triangle meshes. In *ACM Transactions on Graphics (TOG)*, volume 27, page 77. ACM, 2008.

1. Calabi Energy on smooth surfaces

$$\Phi(\mathbf{g}) = \int_S K^2 dA,$$

2. Calabi Flow on smooth surfaces

$$\frac{dg_{ij}}{dt} = 2\Delta K g_{ij}$$

3. Calabi Flow with isometric coordiantes

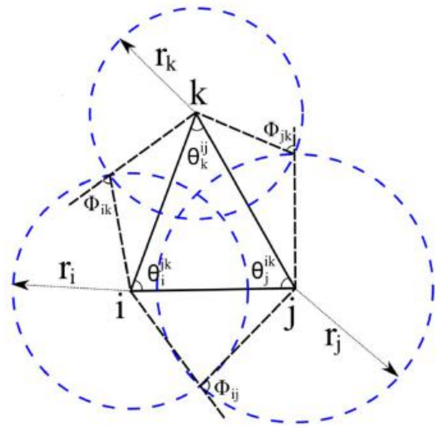
$$\frac{d\lambda}{dt} = \Delta K$$

Riemann metric $\mathbf{g} = e^{2\lambda}\mathbf{g}_0$

Conformal Class

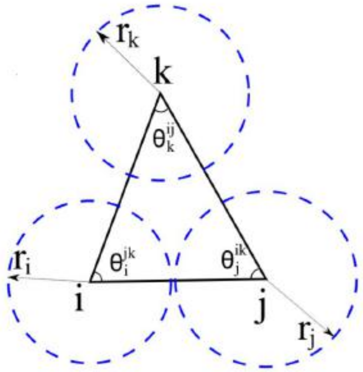
1. Thurston's circle packing metric:

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2 r_i r_j \cos \Phi_{ij}}$$



2. Inversive distance circle packing metric:

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2 r_i r_j I_{ij}}$$



3. If $\{\Phi_{ij}\}$ $\{I_{ij}\}$ and the same, then the metric is in same conformal class

We use inverse distance circle packing in our experiments

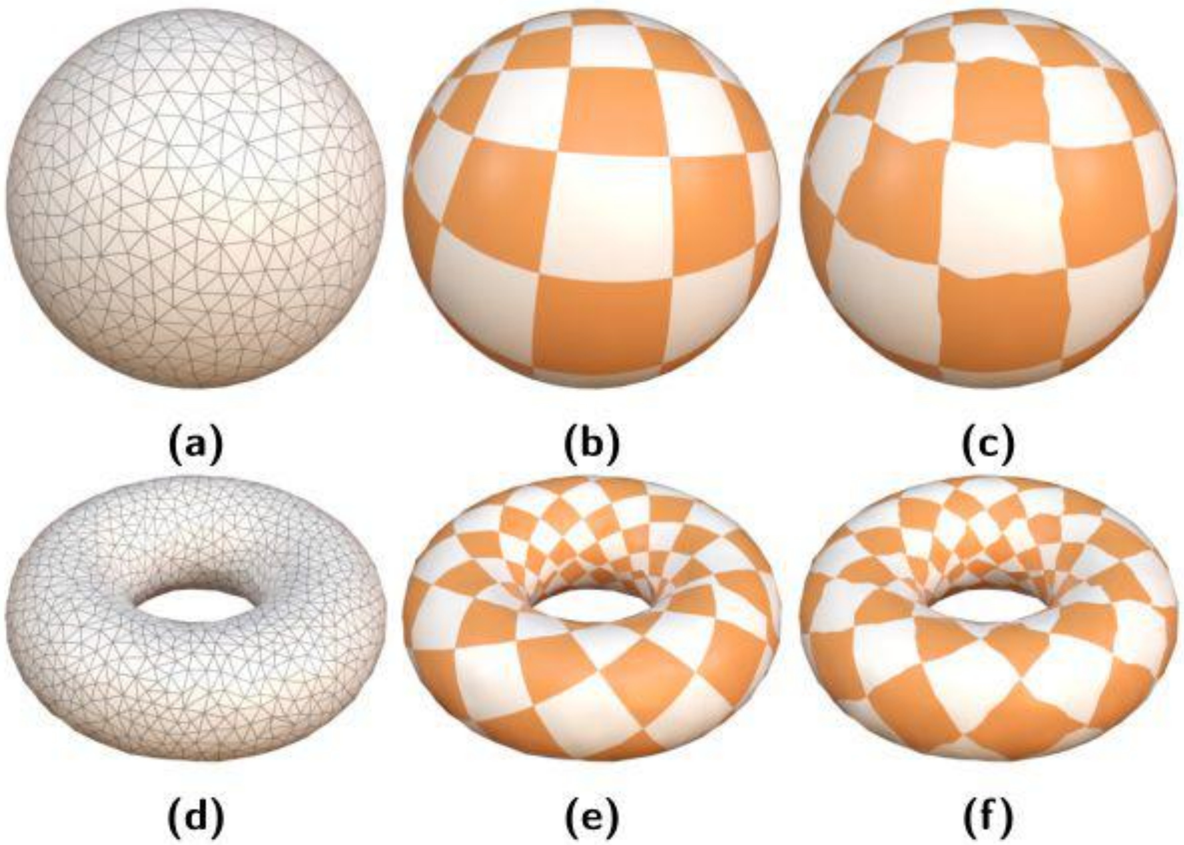


Figure 5: (a), (d) The original meshes; (b), (e) conformal parameterization with inverse distance circle packing metric; (c), (f) conformal parameterization with Thurston's circle packing metric.

1. Discreate Calabi energy on discrete meshes

$$\mathbf{C}(u) = \sum_{v_i \in V} (\bar{K}_i - K_i)^2$$

2. Discrete Calabi Flow on discrete meshes

$$\frac{d\mathbf{u}}{dt} = \Delta_{dual}(\mathbf{K} - \bar{\mathbf{K}}).$$

Discrete Ricci flow

Conformal factor: $u_i = \log r_i$

$$\frac{d\mathbf{u}}{dt} = \mathbf{K} - \bar{\mathbf{K}}$$

Discrete Calabi Flow

Dual Laplacian

$$\frac{d\mathbf{u}}{dt} = \Delta_{dual}(\mathbf{K} - \bar{\mathbf{K}})$$

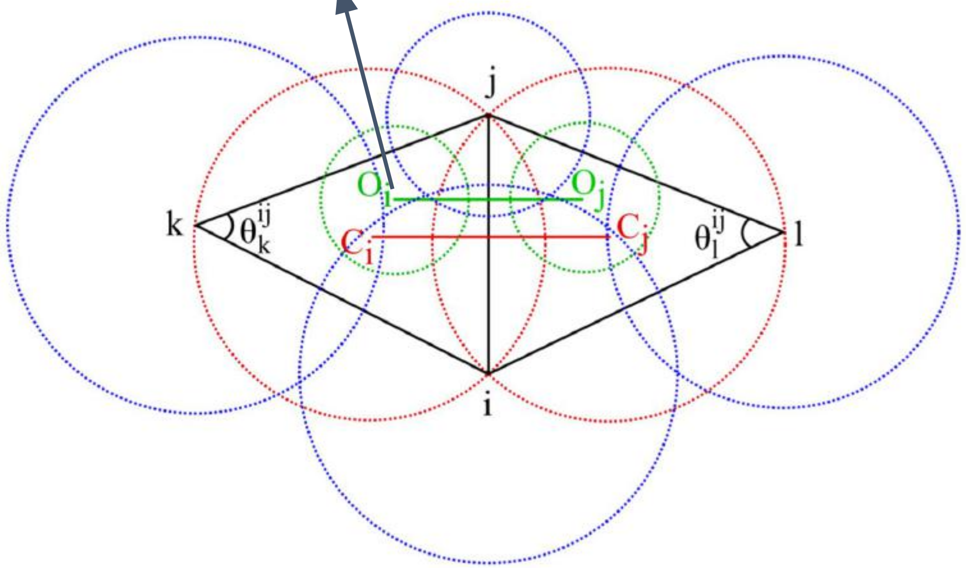
Target curvature

$$\Delta_{dual} = -L_{dual} = -\begin{pmatrix} \frac{\partial K_1}{\partial u_1} & . & . & . & \frac{\partial K_1}{\partial u_N} \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial K_N}{\partial u_1} & . & . & . & \frac{\partial K_N}{\partial u_N} \end{pmatrix}$$

Dual Laplacian is positively defined, so curvature equals to target curvature after convergence.

Computation of Dual Laplacian

Power center



$$\Delta_{dual} = -L_{dual} = -\begin{pmatrix} \frac{\partial K_1}{\partial u_1} & \cdot & \cdot & \cdot & \frac{\partial K_1}{\partial u_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial K_N}{\partial u_1} & \cdot & \cdot & \cdot & \frac{\partial K_N}{\partial u_N} \end{pmatrix}$$

$$(L_{dual})_{ij} = \frac{|O_i O_j|}{l_{ij}}$$

Compare to the cotangent Laplacian:

$$(L_{cot})_{ij} = \frac{\theta_k^{ij} + \theta_l^{ij}}{2} = \frac{|C_i C_j|}{l_{ij}}$$

Power circle : orthogonal to three vertex circles simultaneously

Algorithm: Gradient Descent

Algorithm 2 Calabi Flow

Compute an initial circle packing metric.
Set target curvatures of each vertex.
while $\max_i |K_i - \bar{K}_i| < \epsilon$ **do**
    Calculate curvatures according current metric.
    Calculate dual laplacian L.
    Compute the updating direction $d\mathbf{u} \leftarrow L^T(\bar{\mathbf{K}} - \mathbf{K})$.
    Update conformal factors of each vertex by $\mathbf{u} \leftarrow \mathbf{u} + \delta d\mathbf{u}$.
end while
Embed the mesh to Euclidean plane.

Gradient Descent

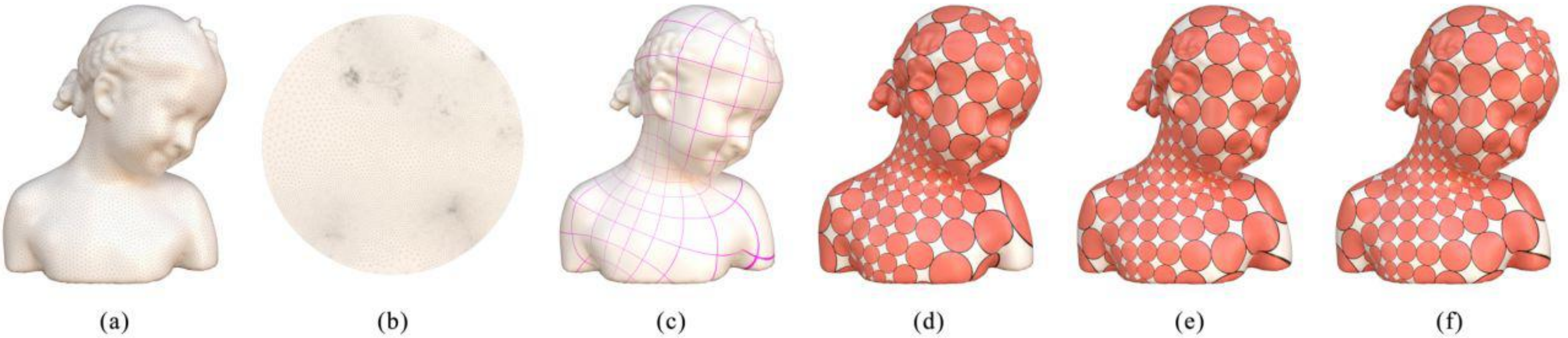


Figure 2: (a) Original mesh; (b) the parameterization with Calabi flow; (c), (d) the texturing with Calabi flow; (e) the texturing with CETM; (f) the texturing with Ricci flow.

1. Fix Boundary

Choose four corners whose target curvatures are pi/2.

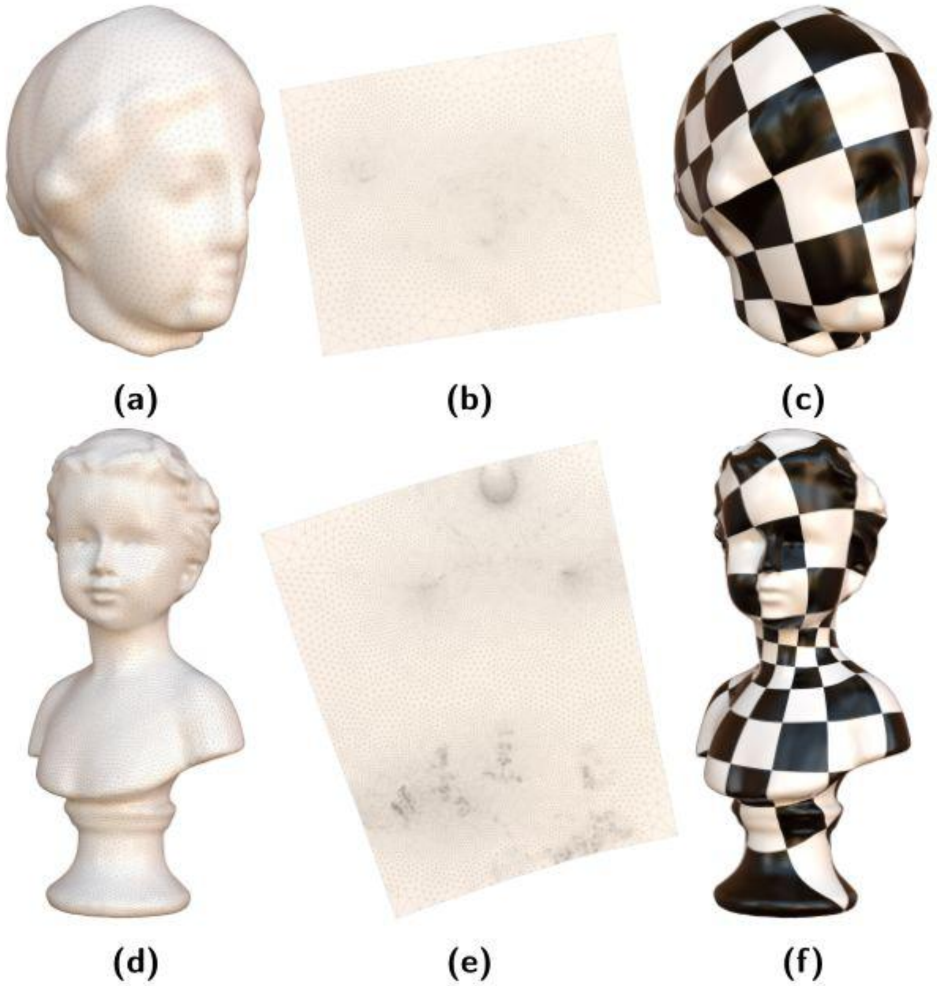
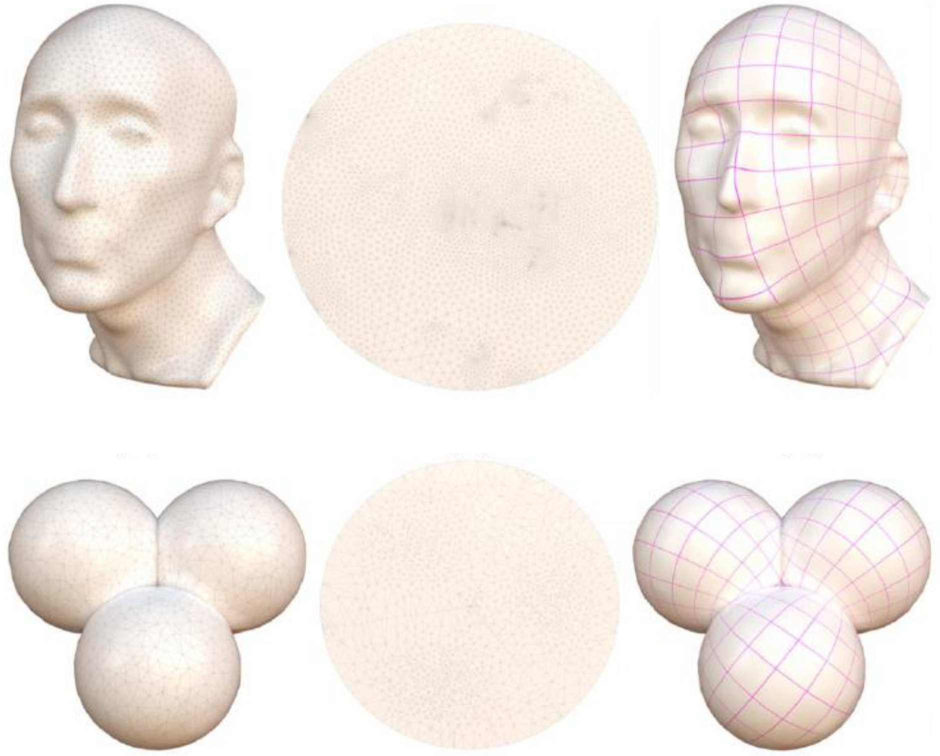


Figure 7: The Calabi flow based parametrizations with fixed rectangle boundary.

2 Circular Boundary

Update target curvatures on the boundary in each iteration:

$$\frac{K_i}{l_{i-1,i}+l_{i,i+1}} \equiv c, \forall v_i \in \partial M$$



3 Free Boundary

Set $d\mathbf{u} = 0$ on the boundary

4 Genus one

(a). Compute a flat metric

(b) Embed mesh by it

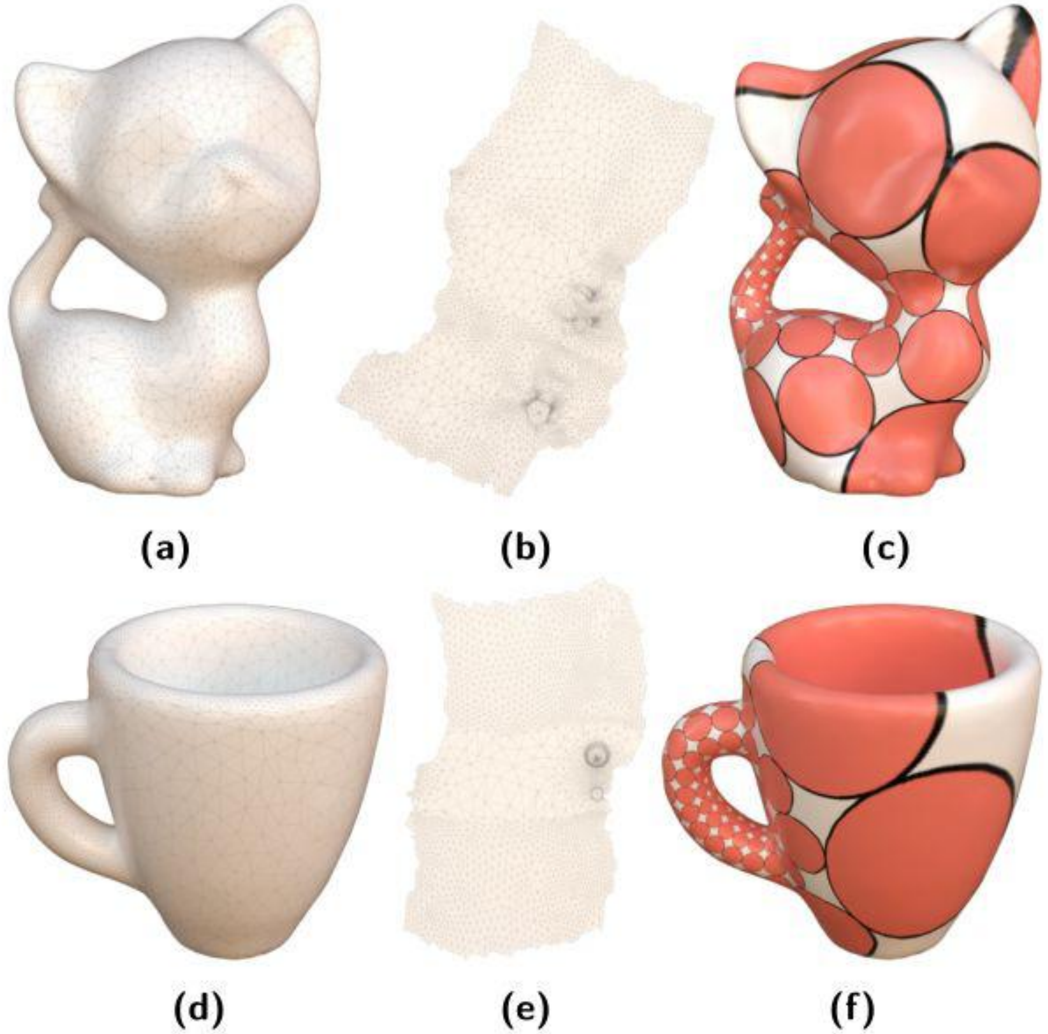


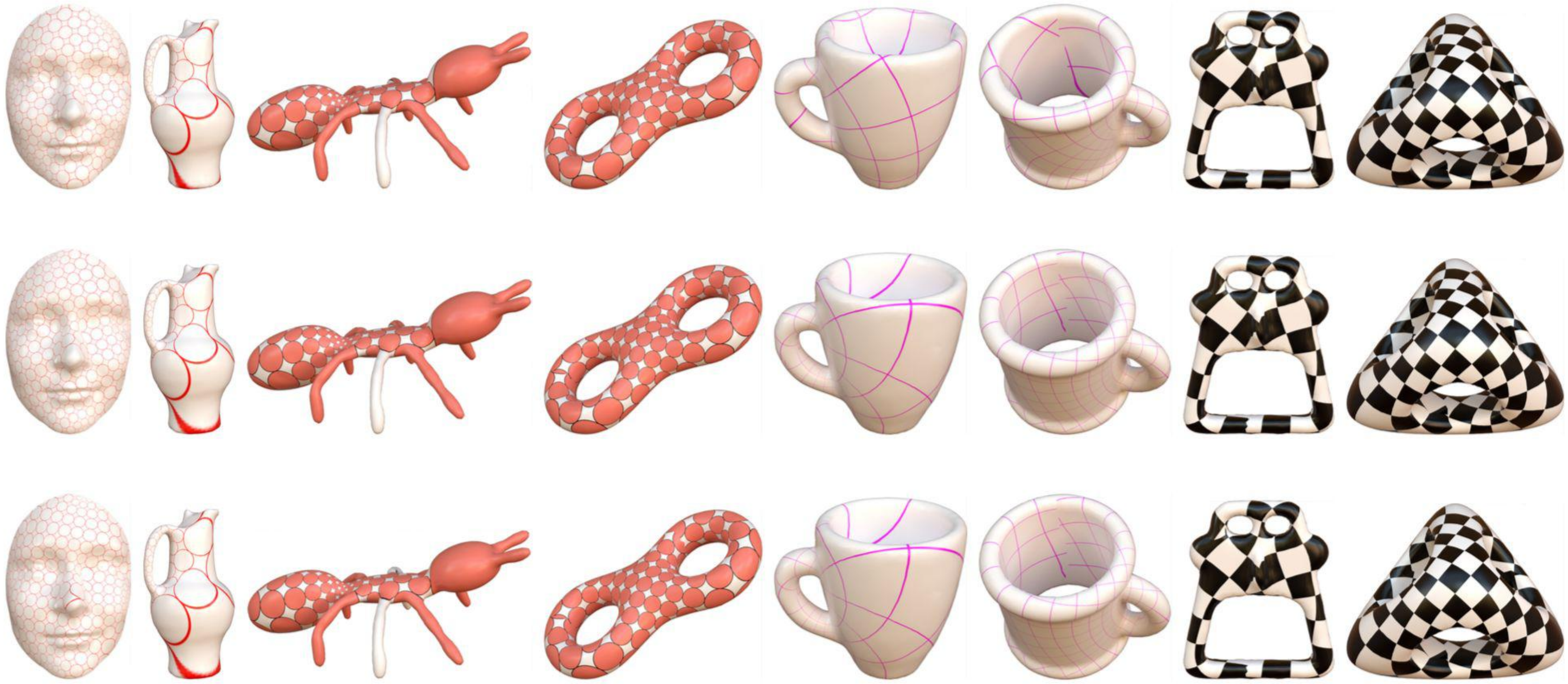
Figure 6: The meshes of genus one and their Calabi flow based parameterizations.

Comparison

Calabi flow:

Ricci flow:

CETM:



As same as Ricci flow and CETM:

The success of discrete Calabi flow depends on the quality of meshes.

Q&A